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**IIR Working Paper WP#13-04**

**March 2013**

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# Interaction-based Foundation of Aggregate Investment Shocks

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March 8, 2013

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**Abstract**

This paper demonstrates that the interactions of firm-level indivisible investments give rise to aggregate fluctuations without aggregate exogenous shocks. I develop a method to derive the distribution of the aggregate capital growth rate by embedding a fictitious tatonnement in a branching process. This method shows that idiosyncratic shocks may lead to non-vanishing aggregate fluctuations when the number of firms tends to infinity. By incorporating this mechanism in a dynamic general equilibrium model with indivisible investment and sticky price, I provide the real business cycle theory with a driver of fluctuations: aggregate investment demand shocks that arise from idiosyncratic productivity shocks. Due to predetermined prices of goods, firms respond to investment shocks by adjusting labor and output, thereby causing the comovements of output and consumption with investment. Numerical simulations show that the model generates aggregate fluctuations comparable to the business cycles in magnitude and correlation structure under standard calibration.

Keywords: Business cycle, strategic complementarity, idiosyncratic shock, law of large numbers, criticality, fat tail.

JEL Classification: E22, E32

# 1 Introduction

One of the most significant questions in macroeconomics is what drives the short-run fluctuations of output in normal times. On the one hand, pursuing this question, researchers have investigated a number of shocks in such fundamental parameters as technology, preference, monetary policy, and expectations. While these aggregate exogenous shocks have been incorporated in the business cycles models and researchers have proven their empirical relevance, there remains a gap between the observed shocks and model interpretation. On the other hand, traditional Keynesians and practitioners in business and policy have stressed the role of autonomous shifts in aggregate investment demand. Nevertheless, the modern business cycles literature has stayed away from the investment demand shocks, since investment demand is an endogenous variable. This paper seeks to augment the literature by bringing these two lines of thought together. It gives a theoretical foundation to the use of investment demand shocks in the standard business cycles model by introducing interactions of firm-level non-linear investment decisions.

This paper shows that the interactions of indivisible investments can generate aggregate demand shocks. An indivisible investment is the simplest example of non-linearity observed in the real economy. When such non-linear behaviors are coupled with each other, the system can generate aggregate fluctuations. However, it is important to note that it only occurs in certain restricted environments. For example, if a pair-wise correlation of non-linear oscillations is too weak, the law of large numbers takes effect and suppresses fluctuations in aggregation. The right amount of correlation is necessary to generate aggregate fluctuations.

To illustrate this point, let us consider a simple reduced model. Suppose that there are $N$ firms, each of which potentially conducts a binary investment. All firms are connected with each other and a firm's investment causes the investment of each one of the other firms with probability $\phi/N$. In this case, the mean number of firms induced to invest by a triggering investment is $\phi$. If a triggering firm induces another firm to invest, the second investment induces another investment in turn. The mean and variance of the total number of firms induced to invest by this process are finite if $\phi < 1$ when $N$ tends to infinity. Thus, the variance of the ratio of investing firms to $N$ scales as $1/N^2$ and quickly becomes negligible. If $\phi > 1$, the chain-reaction process is explosive, and thus, there is a non-negligible chance of all firms investing simultaneously, which does not agree with the observations of business cycles in normal times.

An interesting phenomenon emerges when $\phi = 1$. At this value, the above process stops in a finite step with probability one even in the infinite limit of $N$, whereas the mean and variance of the total number of investing firms diverge. When the number of initial triggering firms follows a binomial distribution with probability $\phi$ and population $N$, it turns out that the ratio of investing firms to $N$ has a non-zero asymptotic variance. This provides a foundation for aggregate investment demand shocks. At the critical value of connectedness $\phi$, micro investment decisions have macro consequences. Then, the central question is why $\phi$ has to have the particular critical value 1 in the real economy. This occurs when the investment threshold for each firm is proportional to aggregate capital. This paper presents a dynamic general equilibrium model in which such a proportional rule emerges naturally.

I consider monopolistic firms competing by producing differentiated goods that are consumed by households with homothetic preference. This economy features aggregate demand externality as in Blanchard and Kiyotaki (1987), with which an increase in aggregate demand proportionally shifts the demand schedule for each good. Under constant returns to scale technology, capital level would be indeterminate in the production sector if capital is adjusted continuously. Now suppose that a capital adjustment is a discrete decision. By incorporating indivisible investments, I obtain two results that do not arise in the case of continuous investments: the capital level given factor prices is locally unique, and the distribution of aggregate investment fluctuations is analytically derived.

Idiosyncratic shocks give rise to aggregate risks under these three conditions. First, the investment decision is non-linear. If the investment response to aggregate capital is smooth and locally linear, then the idiosyncrasies of micro-level investments cancel out with each other as the law of large numbers predicts. Second, the investment decisions complement each other. That is to say, even though the demand for toothbrush is discrete, it does not generate aggregate fluctuations as long as the household toothbrush demands are independent of each other. Third, the complementarity is large. It must be large enough for a firm's investment to induce, on average, one other investment. This implies that the complementarity leads to indeterminacy if the investments are continuous rather than discrete. Considering these restrictive conditions helps us to identify possible loci where interactions pose aggregate risks. There are a few such aggregate phenomena in an economy: one notable example is Keynes' beauty contest of security traders; another

is the pricing of a good given an aggregate price level. This paper proposes firm-level investment decisions as another phenomenon that meets these conditions.

This paper delivers three results. First, I numerically show that the dynamic general equilibrium with indivisible capital, sticky price setting, and many but a finite number of firms generates aggregate fluctuations comparable to the business cycles in their magnitude and correlation structure. Second, an asymptotic distribution function of the aggregate capital fluctuation is derived when the number of firms tends to infinity. The distribution has a heavier tail than the normal distribution.[1] The fat tail indicates that the size of aggregate investment is sensitive to the detailed configuration of firms' positions in the inaction band. This sensitivity to the detailed configuration causes the aggregate investment to exhibit fluctuations in the course of the evolution of capital profile driven by depreciation and discrete investments. Third, I show that the variance of aggregate fluctuations does not vanish at the infinite limit of the number of firms. Even though an economy consists of an infinite number of firms, the non-linear behavior at the firm level does not cancel out in aggregation. This result contrasts to the sectoral models that lack a strong amplification mechanism of idiosyncratic shocks due to the law of large numbers.

[1]Nirei (2006) derived a similar distribution. The present paper differs from the previous one as follows. First, this paper shows the non-zero asymptotic variance of the aggregate growth rate, which was lacking in the previous paper. Second, this paper is worked out in a standard real business cycles framework. Third, this paper eliminates the assumption in the previous paper that the variance of idiosyncratic shocks depends on the number of firms.

I employ a fictitious tatonnement process to derive the distribution of aggregate fluctuation. An investment by a firm increases the aggregate capital and output in the next period. Because of the aggregate demand externality, the higher output induces the other firms to produce more in the next period and thus to invest more in this period. Then, there is a chance of a chain reaction of investments in which one firm's investment triggers another. I formalize this chain reaction as a fictitious best-response dynamics that converges to an equilibrium. The size of the chain reaction depends on the configuration of firms' positions in the inaction band. With a one-sided (S,s) policy, a firm's position in the inaction band asymptotically follows a uniform distribution. Thus, I derive the unconditional distribution of aggregate investment size by drawing a profile of firms' positions from a jointly uniform distribution with its dimension being equal to the number of firms.

Several studies have pointed out the synchronized timing of firms' discrete actions as an important source of macroeconomic fluctuations. Shleifer (1986) demonstrated that the event of synchronized actions can recur deterministically and endogenously through self-fulfilling expectations of periodic adjustments. Jovanovic (1987) posed the question as to when idiosyncratic shocks give rise to aggregate risks. Durlauf (1993) showed that the aggregate size of synchronized actions depends on the detailed configuration of agents' states and can exhibit a long-run path-dependence. I extend this literature by presenting a sharper characterization of the synchronization in a standard business cycle model. I obtain an analytical expression of the fluctuation magnitude with parameters that can be estimated from firm-level data.

Scholars working on interaction-based models and those working on (S,s) economies, independently from each other, have tackled the question of how to analyze the aggregate fluctuations that arise from micro-level discreteness, or more generally, micro-level non-linearity. The models of interactions and non-linear dynamics have shed light on the possibility of endogenous fluctuations arising from micro-level non-linearity, as in Glaeser, Sacerdote, and Scheinkman (1996), Brock and Hommes (1997), and Brock and Durlauf (2001). The (S,s) literature, in contrast, concentrates on macroeconomy where pricing or investment incurs fixed costs and thus exhibits non-linearity at the micro level. Typically, an aggregate (S,s) model features a continuum of firms as in Thomas (2002). This modeling choice precludes the possibility in which interactions of "granular" firms give rise to aggregate fluctuations as in the interaction-based models. While I draw on the (S,s) literature in some important respects, the fluctuation results of this paper are obtained in the model with many but a finite number of firms, and the intuition of the results is analogous to that of the interaction-based models.

This paper contributes to the ongoing debate on the origins of business cycle fluctuations. Researchers such as Fisher (2006) and Justiniano, Primiceri, and Tambalotti (2010) have shown the importance of investment-specific technology shocks in accounting for business cycles in dynamic general equilibrium models. This paper provides a microfoundation for the autonomous fluctuations of aggregate investments. This paper shares its motivation with the literature on sunspot equilibrium, such as Galí (1994) and Wang and Wen (2008), but it differs in that the agents' expectation system is dynamically determinate. In this model, the agents' expectation system is a continuum

approximation of the equilibrium dynamics which is a system of non-linear dynamics of a finite number of firms. Unlike sunspot models, the equilibrium outcome is locally unique due to the discreteness of micro-level decisions. The mechanism for scale-free fluctuations in this model is related to the "break of the law of large numbers" argument in Gabaix (2011) and Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012). In their approach, the break is caused by fat-tailed distributions of firm size or the firms' heterogeneous influence on other firms. This paper complements their findings by demonstrating the aggregate fluctuations even when the firm size and the influence vector are homogeneous. The fluctuation mechanism in this paper is most closely related to the self-organized criticality models (Bak, Chen, Scheinkman, and Woodford (1993)). In those models, an individual action causes an "avalanche" of other actions, and the avalanche size follows a fat-tail distribution. While the preceding self-organized criticality models feature locally interacting firms, this paper is concerned with firms that interact globally (i.e., with all the other firms) in goods markets in a dynamic general equilibrium.

The rest of this paper is organized as follows. Section 2 presents a dynamic general equilibrium model with indivisible capital. I present a model with a continuum of firms and a model with a finite number of firms. The equilibrium dynamics for the former model coincides with the expectation system for the latter model with approximation. By numerically simulating the finite model, I show that equilibrium paths mimic the business cycles in the magnitude of standard deviations and correlations. Section 3 analytically shows that aggregate fluctuations arise in a model without aggregate shocks.

Section 4 discusses the issues on model assumptions and implications. Section 5 concludes. Most proofs are shown in the Appendix, and some detailed derivations are shown in a separate Technical Appendix accompanying this paper.

# 2 Model

In this section, I construct a dynamic general equilibrium model with indivisible investments and predetermined prices of goods. I first present a model with a continuum of firms, and then present a model with a finite number of firms. The equilibrium system of the continuum economy turns out to coincide with the expectation system in the finite economy.

## 2.1 Continuum economy

### 2.1.1 Households

There is a representative household with King-Plosser-Rebelo preference. The representative household maximizes utility

$$\mathrm{E}_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}C_\tau^{1-\sigma}(1-\psi L_\tau^{\zeta})^{1-\sigma}/(1-\sigma)\right]$$

by choosing consumption $C_\tau$ and labor supply $L_\tau$ subject to $C_\tau = w_\tau L_\tau + D_\tau$, $\forall\tau$. $D_t$ denotes aggregate dividends that the household receives from firms. Each firm $i$ owns capital and delivers dividend $d_{i,t}$. Households as shareholders instruct each firm to maximize its expected discounted sum of dividends stream $\mathrm{E}_t\left[\sum_{\tau=t}^{\infty}\Delta_{t,\tau}d_{i,t}\right]$. The discount factor is $\Delta_{t,\tau}\equiv\Pi_{s=t+1}^{\tau}R_s^{-1}$,

where $\Delta_{t,t} = 1$ by convention, and $R_t$ is the inverse of a stochastic discount factor

$$R_t^{-1} \equiv \beta C_t^{-\sigma}(1-\psi L_t^{\zeta})^{1-\sigma}/(C_{t-1}^{-\sigma}(1-\psi L_{t-1}^{\zeta})^{1-\sigma}). \tag{1}$$

The representative household consumes a composite consumption good that is produced by a CES function $C_t = (\int z_{c,i,t}^{(\eta-1)/\eta} di)^{\eta/(\eta-1)}$, where $\eta > 1$ is the elasticity of substitution. Let $p_{i,t}$ denote the price of good $i$, and let the aggregate price $(\int p_{i,t}^{1-\eta} di)^{1/(1-\eta)}$ be normalized to 1. A household's cost minimization yields the demand function for $z_{c,i,t}$, which satisfies $\int p_{i,t} z_{c,i,t} di = C_t$. The first-order conditions with respect to $C_t$ and $L_t$ yield:

$$w_t = C_t \psi \zeta L_t^{\zeta-1}/(1-\psi L_t^{\zeta}). \tag{2}$$

### 2.1.2 Firms

In this section, I suppose that there are a continuum of firms indexed by $i \in [0,1]$. Each firm has a Cobb-Douglas production function with constant returns to scale: $y_{i,t} = a_{i,t} k_{i,t}^{\alpha} l_{i,t}^{1-\alpha}$. Productivity $a_{i,t}$ is a random variable, independently and identically distributed across $i$ and $t$. I assume that each firm knows the realization of $a_{i,t}$ in period $t-1$. I also assume that the support of $\log a_{i,t}$ is bounded so that $\Pr(|\log a_{i,t} - \log a_{i,t-1}| < |\log(1-\delta)|\alpha(1/\rho - 1)) = 1$, where $\delta$ and $\rho$ are parameters as defined below.

Firm $i$ produces good $i$ monopolistically, and it faces demand function $y_{i,t} = p_{i,t}^{-\eta} Y_t$, where $Y_t \equiv (\int y_{i,t}^{(\eta-1)/\eta} di)^{\eta/(\eta-1)}$ denotes the aggregate output. Firm $i$ owns physical capital $k_{i,t}$, which accumulates as $k_{i,t+1} = (1-\delta)k_{i,t} + x_{i,t}$. The investment good is produced by the CES function similar to the consumption good: $x_{i,t} = (\int z_{i,j,t}^{(\eta-1)/\eta} dj)^{\eta/(\eta-1)}$. The dividend paid by $i$ is $d_{i,t} = p_{i,t} y_{i,t} - w_t l_{i,t} - x_{i,t}$. The aggregate dividend is $D_t = \int d_{i,t} di$.

I assume that the firm's capital choice is restricted to be binary: $k_{i,t+1} \in \{(1-\delta)k_{i,t}, \lambda_i(1-\delta)k_{i,t}\}$, where $\lambda_i(1-\delta) > 1$. Capital $k_{i,t+1}$ has to be either at the depreciated level $(1-\delta)k_{i,t}$ or the depreciated level multiplied by the firm-specific indivisibility parameter $\lambda_i$. This binary constraint is equivalent to assuming that the firm can choose gross investment rate $x_{i,t}/k_{i,t}$ only either at 0 or $(\lambda_i - 1)(1-\delta)$, namely, inaction or lumpy investment, respectively. This constraint reflects the indivisibility of physical investment such as equipment and structure. It can also be interpreted as a shortcut for modeling the lumpy behavior, which typically occurs as an optimal investment policy under fixed adjustment costs. This paper is concerned with the aggregate consequence of the non-linear behavior of firms induced by this indivisibility.

I assume that firm $i$ commits to the price of its product $p_{i,t}$ one period ahead. Namely, firm $i$ decides $p_{i,t}$ in period $t-1$. Once the demand for $y_{i,t}$ is realized in $t$, firm $i$ meets the demand by adjusting labor hours $l_{i,t}$. Thus, firm $i$'s problem in period $t-1$ is to maximize $\mathrm{E}_{t-1}\left[\sum_{\tau=t-1}^{\infty} \Delta_{t-1,\tau} d_{i,\tau}\right]$ by choosing $p_{i,t}$ and $k_{i,t}$ subject to the demand function, production function, and binary constraint for capital. The optimal price is solved as $p_{i,t} = (a_{i,t}^{1/\alpha} k_{i,t}/K_t)^{-\alpha/(\eta(1-c_1))}$, where $c_1 \equiv (1-1/\eta)(1-\alpha)$, $\rho \equiv \alpha(1-1/\eta)/(1-c_1)$, and $K_t \equiv (\int a_{i,t}^{\rho/\alpha} k_{i,t}^{\rho}\, di)^{1/\rho}$. Substituting $p_{i,t}$ in the demand function and aggregating across $i$, I obtain

$$K_t = \left(\mathrm{E}_{t-1}[(w_t/c_1) Y_t^{1/(1-\alpha)}/R_t]/\mathrm{E}_{t-1}[Y_t/R_t]\right)^{(1-\alpha)/\alpha}. \tag{3}$$

By using the optimal price, firm $i$'s problem in period $t-1$ boils down to choosing $k_{i,t}$ from a binary set to maximize $\pi(k_{i,t}) = (1-c_1)\mathrm{E}_{t-1}[Y_t/R_t] a_{i,t}^{\rho/\alpha} (k_{i,t}/K_t)^{\rho} - (1-(1-\delta)\mathrm{E}_{t-1}[R_t^{-1}])k_{i,t}$. The optimal strategy for firm $i$ is to invest in $t-1$ when $(1-\delta)k_{i,t-1}$ is below a threshold level $k_{i,t}^*$, and not to invest otherwise.

At the threshold, the firm must be indifferent between investing and not investing. This implies $\pi(\lambda_i k^*_{i,t}) = \pi(k^*_{i,t})$, because the capital stream after $t+1$ when investing in $t-1$ coincides with that when the investment is deferred to $t$. Threshold $k^*_{i,t}$ is uniquely and explicitly obtained by solving this equation, as $\pi$ is strictly concave since $\rho < 1$.

### 2.1.3 Market-clearing conditions

Consumption $C_t$ and investment $x_{i,t}$ are composite goods, and the derived demand for good $j$ is denoted by $z_{c,j}$ for a household and $z_{i,j}$ for firm $i$. Thus, the goods markets clear by $y_{j,t} = z_{c,j,t} + \int z_{i,j,t} di$, $\forall j$. Aggregating these under cost-minimization conditions, I obtain

$$Y_t = C_t + X_t, \tag{4}$$

where $X_t \equiv \int x_{i,t} di$ is aggregate investment.

The labor market clearing condition is $L_t = \int l_{i,t} di$. By substituting the price-setting rule in labor demand $l_{i,t} = (p_{i,t}^{-\eta} Y_t / (a_{i,t} k_{i,t}^{\alpha}))^{1/(1-\alpha)}$ and aggregating, I obtain an aggregate production function:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}. \tag{5}$$

### 2.1.4 Capital gap distribution

I define capital gap as the difference between actual and threshold capital normalized by lumpiness: $s_{i,t} \equiv (\log k_{i,t} - \log k^*_{i,t}) / \log \lambda_i$. This gap $s_{i,t}$ always takes a value between 0 and 1 at equilibrium. Using $s_{i,t}$, the aggregate capital can be written as $K_t = (\int a_{i,t}^{\rho/\alpha} \lambda_i^{\rho s_{i,t}} k^{*\rho}_{i,t} di)^{1/\rho}$. Using this, I obtain a marginal

cost condition and a threshold rule as follows:

$$1 = \left( \int a_{i,t}^{\frac{\rho}{\alpha(1-\rho)}} \left( \frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\frac{\rho}{1-\rho}} \lambda_i^{\rho s_{i,t}} di \right)^{\frac{1-\rho}{\rho}} B_{t-1}, \tag{6}$$

$$B_{t-1} \equiv \frac{(1-c_1)\mathrm{E}_{t-1}[Y_t/R_t]^{\frac{1}{\alpha}}}{(1-(1-\delta)\mathrm{E}_{t-1}[R_t^{-1}])\mathrm{E}_{t-1}[(w_t/c_1)Y_t^{\frac{1}{1-\alpha}}/R_t]^{\frac{1-\alpha}{\alpha}}}, \tag{7}$$

$$k_{i,t}^* = b_{i,t} K_t, \tag{8}$$

$$b_{i,t} \equiv B_{t-1} \left( a_{i,t}^{\rho/\alpha} (\lambda_i^\rho - 1)/(\lambda_i - 1) \right)^{1/(1-\rho)}. \tag{9}$$

The capital gap develops as

$$s_{i,t+1} = \left( \frac{\log(1-\delta) + \log k_{i,t}^* - \log k_{i,t+1}^*}{\log \lambda_i} + s_{i,t} + 1 \right) \bmod 1, \tag{10}$$

where "$x$ mod 1" denotes the remainder after division of $x$ by 1. Starting from an initial state $s_{i,0}$, $s_{i,t}$ is given as natural depreciation $t \log(1-\delta)$ divided by $\log \lambda_i$, plus a random variable, and taken modulo 1. When $1/\log \lambda_i$ has a density, this remainder converges to a uniform distribution on a unit interval (Engel, 1992, 3.1.1).

**Proposition 1** *As $t \to \infty$, $s_{i,t}$ converges in distribution to a uniform random variable in $[0, 1)$.*

The proof is in Appendix A. Proposition 1 corresponds to a robust feature of one-sided (S,s) economies as shown in Caplin and Spulber (1987) and Caballero and Engel (1991). Note that the cross-section distribution of $s_{i,t}$ stays at the uniform distribution even if aggregate variables fluctuate, since a shift in $K_t$ merely rotates the distribution of $s_{i,t}$ on a circle of unit circumference.

### 2.1.5 Aggregate variables under a stationary gap distribution

Since there are a continuum of firms, idiosyncratic shocks $a_{i,t}$ are aggregated out and there is no aggregate risk in this economy. Thus, (3) is reduced to

$$K_t = (w_t/c_1)^{\frac{1-\alpha}{\alpha}} Y_t. \tag{11}$$

I consider the case of a stationary gap distribution. Substituting the uniform distribution of $s_{i,t}$ in (6) and using (11), I obtain a familiar condition on marginal costs in the constant returns to scale economy:

$$1 = a^{\rho-1}(1-c_1)(w_t/c_1)^{-\frac{1-\alpha}{\alpha}}(R_t - 1 + \delta)^{-1}, \tag{12}$$

$$a \equiv \left( \int \left( \frac{\lambda_i^\rho - 1}{(\lambda_i - 1)^\rho} \right)^{\frac{1}{1-\rho}} \frac{a_{i,t}^{\frac{\rho}{\alpha(1-\rho)}}}{\rho \log \lambda_i} di \right)^{-\frac{1}{\rho}}. \tag{13}$$

Under the stationary distribution, the threshold becomes a function of only the idiosyncratic productivity and the aggregate capital:

$$k_{i,t}^*/K_t = b_{i,t} = a \left( a_{i,t}^{\rho/\alpha} (\lambda_i^\rho - 1)/(\lambda_i - 1) \right)^{1/(1-\rho)}. \tag{14}$$

The threshold capital $k_{i,t}^*$ can be translated to threshold gap $s_{i,t}^*$, where firms with $s_{i,t} \in [0, s_{i,t}^*)$ invest in $t$. Since $a_{i,t+1}$ is known to $i$ in $t$, $s_{i,t+1} = 0$ holds at $s_{i,t} = s_{i,t}^*$. Thus, the threshold is obtained from (10) as

$$s_{i,t}^* = \frac{\log k_{i,t+1}^* - \log k_{i,t}^*}{\log \lambda_i} - \frac{\log(1-\delta)}{\log \lambda_i}. \tag{15}$$

Due to the assumption of bounded increment of $\log a_{i,t}$, gap $s_{i,t}$ always decreases over time unless there is an upward jump by 1. Aggregate gross investment under the stationary uniform distribution of $s_{i,t}$ is then written

as follows:[2]

$$X_t = \int \int_0^{s^*_{i,t}} (\lambda_i - 1)(1-\delta)\lambda_i^{s_{i,t}} k^*_{i,t} ds_{i,t}\, di = \rho a^{1-\rho}(K_{t+1} - (1-\delta)K_t). \quad (16)$$

### 2.1.6 Equilibrium

I consider an economy in which capital gap $s_{i,0}$ has achieved a stationary uniform distribution across $i$. An equilibrium consists of pricing functions $w(K)$ and $R(K)$, the law of motion for $K$, and aggregate allocation $(Y, X, C, L, D)$ such that the allocation solves the household's problem given prices, that the law of motion and the allocation are consistent with the firms' optimal investment policy, and that the goods and labor markets clear. The equilibrium path satisfies the system of equations (1, 2, 4, 5, 6, 11, 16). Bar denotes the steady-state values. By log-linearizing the system around the steady state, it is shown that the equilibrium path is locally determinate under a mild condition.[3]

**Proposition 2** *There exists a unique saddle point path for the log-linearized system of (1, 2, 4, 5, 6, 11, 16), if* $\bar{X}/\bar{Y} \leq \alpha$ *holds.*

## 2.2 Finite economy

In this section, I turn to an economy where there are many but finite $N$ firms instead of a continuum of firms. The economy experiences some fluctuations due to finite idiosyncratic shocks. I will show that the fluctuation of aggregate investment $X_t$ remains non-trivial even when $N$ is large. As before, I employ

[2]See the Technical Appendix for a detailed derivation.

[3]The proof is standard and provided in the Technical Appendix.

a sticky price assumption, under which firm $i$ commits itself to meeting the demand in $t$ at price $p_{i,t}$ that is decided in period $t-1$. When $X_t$ differs from the expected level due to finite shocks, firms adjust their labor demand, and the labor market clears by adjusting the nominal wage. Thus, under predetermined prices, the investment shock causes quantity adjustments in hours worked, production, and consumption.

Aggregate variables are now redefined by averages such as $Y_t \equiv (\sum_{i=1}^N y_{i,t}^{(\eta-1)/\eta}/N)^{\eta/(\eta-1)}$, $C_t = (\sum_{i=1}^N z_{c,i,t}^{(\eta-1)/\eta}/N)^{\eta/(\eta-1)}$, $X_t \equiv \sum_{i=1}^N x_{i,t}/N$, $K_t \equiv (\sum_{i=1}^N a_{i,t}^{\rho/\alpha} k_{i,t}^{\rho}/N)^{1/\rho}$, and $P_t \equiv (\sum_{i=1}^N p_{i,t}^{1-\eta}/N)^{1/(1-\eta)} (= 1)$. The labor market clearing condition is $L_t = \sum_{i=1}^N l_{i,t}/N$. Similar to the continuum case, equilibrium conditions are derived as (1, 2, 3, 4, 5, 7, 8) and

$$K_{t+1} = \left( \sum_{i:(1-\delta)k_{i,t} < k^*_{i,t+1}} \frac{(\lambda_i(1-\delta)k_{i,t})^\rho}{N} + \sum_{i:(1-\delta)k_{i,t} \geq k^*_{i,t+1}} \frac{((1-\delta)k_{i,t})^\rho}{N} \right)^{\frac{1}{\rho}}, \tag{17}$$

$$X_t \equiv \sum_{i:(1-\delta)k_{i,t} < k^*_{i,t+1}} (\lambda_i - 1)(1-\delta)k_{i,t}, \tag{18}$$

$$1 = \left( \sum_{i=1}^N a_{i,t}^{\frac{\rho}{\alpha(1-\rho)}} \left( \frac{\lambda_i^\rho - 1}{\lambda_i - 1} \right)^{\frac{\rho}{1-\rho}} \frac{\lambda_i^{\rho s_{i,t}}}{N} \right)^{\frac{1-\rho}{\rho}} B_{t-1}. \tag{19}$$

The state space involves the distribution of gap $s_{i,t}$, which is included in the information set for the conditional expectation in period $t$ and affects the summations in (17–19). The equilibrium is similar to that of Krusell and Smith, Jr. (1998) and is difficult to solve exactly. Thus, I approximate the equilibrium system by using the stationary distributions of $s_{i,t}$ and $a_{i,t}$ with a continuum of firms. By this approximation, the summations across $i$ in (17–19) are replaced with integrals over the uniform distribution of $s_{i,t}$. I

assume that agents use this approximated equilibrium system to form expectations of future variables, whereas the exact realizations of $k^*_{i,t+1}, K_{t+1}, X_t$ are determined by (8, 17, 18) while keeping summations. Then, the system of equations for the agents' forecast becomes (1, 2, 3, 4, 5) and

$$1 = \frac{a^{\rho-1}(1-c_1)\mathrm{E}_{t-1}[Y_t/R_t]^{\frac{1}{\alpha}}}{(1-(1-\delta)\mathrm{E}_{t-1}[R_t^{-1}])\mathrm{E}_{t-1}[(w_t/c_1)Y_t^{\frac{1}{1-\alpha}}/R_t]^{\frac{1-\alpha}{\alpha}}}, \tag{20}$$

$$K^e_{t+1} = (1-\delta)K_t + (a^{\rho-1}/\rho)X^e_t, \tag{21}$$

$$X_t = X^e_t e^{\epsilon_t}, \tag{22}$$

$$K_{t+1} = (1-\delta)K_t + (a^{\rho-1}/\rho)X_t = K^e_{t+1} + (a^{\rho-1}/\rho)X^e_t(e^{\epsilon_t}-1). \tag{23}$$

The aggregate investment demand shock $\epsilon_t$ enters (22), representing the difference between realized and expected investments.

The expectation system (1, 2, 3, 4, 5, 20, 21, 22, 23) is approximated in the first order as follows. Let tilde denote log-difference from the steady state. Following Sims (2001), for the log-difference variables, time-subscripts indicate the period in which the variable is observable to agents. For example, a predetermined variable $K_t$ corresponds to $\tilde{K}_{-1}$, while $E_{t-1}C_t$ corresponds

to $E_{-1}\tilde{C}_0$. The log-linearized expectation system is

$$\begin{aligned}
\tilde{K}_0 &= (1-\delta)\tilde{K}_{-1} + \delta\tilde{X}_0, \\
\tilde{Y}_0 &= (\bar{C}/\bar{Y})\tilde{C}_0 + (\bar{X}/\bar{Y})\tilde{X}_0, \\
\tilde{Y}_0 &= \alpha\tilde{K}_{-1} + (1-\alpha)\tilde{L}_0, \\
\mathrm{E}_{-1}\tilde{Y}_0 &= \tilde{K}_{-1} - \frac{1-\alpha}{\alpha}\mathrm{E}_{-1}\tilde{w}_0, \\
\tilde{w}_0 &= \tilde{C}_0 + \left(\zeta - 1 + \bar{w}\bar{L}/\bar{C}\right)\tilde{L}_0, \\
0 &= \frac{1-\alpha}{\alpha}\mathrm{E}_{-1}\tilde{w}_0 + \frac{\bar{R}}{\bar{R}-1+\delta}\mathrm{E}_{-1}\tilde{R}_0, \\
\tilde{R}_0 &= \sigma(\tilde{C}_0 - \tilde{C}_{-1}) - (\sigma-1)(\bar{w}\bar{L}/\bar{C})(\tilde{L}_0 - \tilde{L}_{-1}), \\
\tilde{X}_0 &= \mathrm{E}_{-1}\tilde{X}_0 + \epsilon_0, \\
\tilde{C}_0 &= \mathrm{E}_{-1}\tilde{C}_0 + \eta_0^C, \quad \tilde{L}_0 = \mathrm{E}_{-1}\tilde{L}_0 + \eta_0^L, \\
\tilde{Y}_0 &= \mathrm{E}_{-1}\tilde{Y}_0 + \eta_0^Y, \quad \tilde{w}_0 = \mathrm{E}_{-1}\tilde{w}_0 + \eta_0^w,
\end{aligned}$$

where $(\eta_0^C, \eta_0^L, \eta_0^Y, \eta_0^w)$ are expectation errors caused by the expectation error in investment, $\epsilon_0$.

The certainty equivalence of the log-linearized expectation system coincides with the log-linearized equilibrium system of a continuum economy. Thus, the expectation system has a determinate solution. Combined with $\epsilon_t$, the equilibrium path fluctuates around the determinate saddle point path.

**Proposition 3** *There exists a unique saddle point path for the expectation system if* $\bar{X}/\bar{Y} \leq \alpha$ *holds.*

The proof is provided in the Technical Appendix.

## 2.3 Investment demand shock

In a finite economy, the investment demand shock $\epsilon_t$ is defined as a log-difference between realized aggregate investment $X_t$ and expected aggregate investment $\mathrm{E}_{t-1}X_t$. $X_t$ is determined along with $K_{t+1}$ and $k^*_{i,t+1}$ by (8, 17, 18) given exact capital $k_{i,t}$ and realized productivity $a_{i,t+1}$. $\mathrm{E}_{t-1}X_t$ is determined by the expectation system (1, 2, 3, 4, 5, 20, 21, 22, 23) given $K_t$. The deviation of actual aggregate investment from the expected value is caused by idiosyncratic productivity shocks $a_{i,t+1}$ for a finite number of firms and the deviation of the gap distribution from the uniform distribution.

Due to the non-linear decision of $k_{i,t+1}$ with strategic complementarity across $i$, there can be multiple solutions for (8, 17) for each state $(k_{i,t}, a_{i,t+1})_i$. For those cases, I set an equilibrium selection rule that picks the solution that minimizes $|\epsilon_t|$ among all the solutions. Namely, this selection rule picks the equilibrium path that minimizes the deviation from the expected equilibrium path determined by the continuum counterpart. In numerical simulations, $\epsilon_t$ is computed as follows. First, $\epsilon_t$ is set to 0, and $K_{t+1}$ and $k^*_{i,t+1}$ are computed given $k_{i,t}$. If $X_t$ under the threshold $k^*_{i,t+1}$ coincides with $\mathrm{E}_{t-1}X_t$, then $\epsilon_t$ is determined at 0. Otherwise, $\epsilon_t$ is adjusted slightly, and the above procedure is repeated until the selected outcome is obtained.

## 2.4 Calibration and numerical simulations

For a benchmark calibration, I set the unit of time as a quarter. The parameters for technology and preference are set at standard values as in Table 1. Firms' markup rate $1/(\eta - 1)$ is set at 10%. The capital intensity $\alpha$ is

set such that the labor share $\bar{w}\bar{L}/\bar{Y}$ is equal to 0.67. The annual rate of depreciation is set at 8% and the annual risk free rate at 4%. The disutility from labor is specified as a quadratic function. Indivisibility parameter $\lambda_i$ is a random variable drawn in period 0 and fixed for later periods. I set that $\lambda_i$ is drawn from a normal distribution with mean 1.028 and standard deviation 0.004 truncated at two standard deviations. I choose this specification to match with the 2.8% plant Herfindahl index estimated by Ellison and Glaeser (1997). Plant Herfindahl measures the representative share of a plant's employment in an industry. When capital size is adjusted by changing the number of plants, plant Herfindahl can be interpreted as a lower bound of capital indivisibility, which coincides with firm-level capital indivisibility if the industry is a monopoly. These parameters and steady state values for the benchmark specification are summarized in Table 1.

The number of firms $N$ is set at 350000 to match with the number of operating manufacturing plants in the US (Cooper, Haltiwanger, and Power (1999)). The logarithm of the idiosyncratic productivity $\log a_{i,t}$ is assumed to follow a normal distribution with standard deviation 0.05%. The mean productivity is set such that the mean of $a_{i,t}^{\rho/(\alpha(1-\rho))}$ (which appears in the threshold rule (8)) is normalized to 1. In the initial period, $s_{i,0}$ is randomly drawn from a uniform distribution, and in each period, productivity $a_{i,t}$ is drawn independently. The equilibrium path is simulated for 1100 periods, from which the first 100 periods are discarded. Table 2 reports the standard deviations and comovement structure of output, consumption, investment, hours worked, and capital.[4] As can be seen, the model is able to generate ag-

[4]The reported moments are averages of 10 simulated runs. Figures in parentheses are

gregate investment fluctuations to the magnitude comparable to the business cycles.

| $\alpha$ | $\delta$ | $\sigma$ | $\beta$ | $\eta$ | $\psi$ | $\zeta$ | $\mathrm{E}(\lambda_i)$ | $\bar{w}\bar{L}/\bar{Y}$ | $\bar{R}$ | $\bar{C}/\bar{Y}$ | $\bar{K}/\bar{Y}$ | $1-\psi\bar{L}^{\zeta}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.26 | 0.02 | 1.5 | 0.99 | 11 | 1 | 2 | 1.028 | 0.67 | 1.01 | 0.84 | 7.74 | 0.72 |

Table 1: Benchmark calibration and endogenous steady-state values

| | Standard deviation (%) | | | | | Correlation with $\tilde{Y}$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\tilde{Y}$ | $\tilde{C}$ | $\tilde{X}$ | $\tilde{L}$ | $\tilde{K}$ | $\tilde{C}$ | $\tilde{X}$ | $\tilde{L}$ | $\tilde{K}$ |
| Benchmark | 2.23 | 0.85 | 6.44 | 3.02 | 0.26 | 0.820 | 0.979 | 0.793 | -0.020 |
| | (0.06) | (0.02) | (0.17) | (0.08) | (0.01) | (0.004) | (0.001) | (0.004) | (0.005) |
| $\sigma = 3$ | 3.48 | 2.83 | 6.42 | 5.26 | 0.25 | 0.928 | 0.900 | 0.924 | -0.171 |
| $\mathrm{E}[\lambda_i] = 1.056$ | 4.23 | 1.59 | 12.21 | 5.62 | 0.50 | 0.828 | 0.978 | 0.800 | -0.011 |
| $N = 100000$ | 2.18 | 0.83 | 6.30 | 2.95 | 0.23 | 0.818 | 0.979 | 0.791 | -0.032 |

Table 2: Standard deviations and correlations of key business cycles variables

The fluctuations in aggregate variables are driven mostly by investment demand shocks $\epsilon_t$, while movements in capital play little role. The investment demand shock $\epsilon_t$ propagates to other variables in two paths: $K_{t+1}$ and $Y_t$. On one hand, an investment demand shock generates an exogenous increase in future capital $K_{t+1}$. This raises future labor productivity and real wage. The prospect of increased labor productivity induces households to consume more in both the next period and the current period. This effect can be seen in the saddle point path in which the marginal utility of consumption is

standard errors.

negatively related to capital. On the other hand, an increase in investment demand raises aggregate demand for contemporaneous goods, provided that consumption demand is unaffected. Firms respond to the increased demand by increasing labor demand, which raises real wage. Households respond to the higher real wage by raising hours worked, which in turn raises the marginal utility of consumption when $\sigma > 1$. Thus, in order to keep the marginal utility lower so that it is on the saddle point path, consumption demand must increase. Hence, the investment demand shock raises consumption, and thus, output. In Table 2, I observe that the standard deviation of consumption relative to investment is larger when $\sigma$ is greater. This result is consistent with the propagation mechanism described above, because the hours-consumption complementarity, given a fixed marginal utility of consumption, becomes larger when $\sigma - 1$ is greater.

# 3 Analytical Results

## 3.1 Random gap distribution

At the heart of the aggregate fluctuations arising from idiosyncratic shocks in the simulations lay the non-linearity and complementarity of firm-level investment decisions. The capital decision $k_{i,t+1}$ is non-linear because of indivisibility and threshold policy. Average capital level $K_{t+1}$ affects threshold $k^*_{i,t+1}$ linearly, but it may or may not induce an adjustment of capital $k_{i,t+1}$. Individual capital is insensitive to a small perturbation in average capital, while it synchronizes with average capital if the perturbation is large.

In this section, I analytically derive the distribution of the aggregate

capital fluctuations. To achieve this, I draw $s_{i,t}$ from a stationary distribution uniform over $[0, 1)$. Similar to Proposition 1 in the continuum economy, $s_{i,t}$ in the finite economy converges to a uniform distribution as $t \to \infty$, independent across $i$. This implies that the probability of drawing a particular profile $(s_{i,t})_i$ from an $N$-dimensional jointly uniform distribution corresponds to the likelihood of the profile realizing in the course of gap profile evolution in a far future.

## 3.2 Equilibrium selection

For each realization of gap and productivity profile $(s_{i,t}, a_{i,t+1})_i$, and given the expected aggregate capital $K^e_{t+1}$, capital profile in the next period $(k_{i,t+1})_i$ is determined by (8, 17). Thus, the distribution function of the growth of $K_t$ is determined by the joint distribution function of $(s_{i,t}, a_{i,t+1})$, if this mapping is one-to-one. In case where there are multiple solutions for (8, 17), I selected the outcome in numerical simulations in Section 2.4 by the following criterion.

**Equilibrium Selection 1** *For each initial capital vector $(k_{i,t})$, pick the equilibrium aggregate capital $K_{t+1}$ that attains the minimum of $|\log K_{t+1} - \log K^e_{t+1}|$ among all $K_{t+1}$ that solve (8, 17).*

By this mechanism, I construct the least-possible volatile fluctuations of aggregate capital in equilibrium. To facilitate the analysis of this equilibrium, I define another selection mechanism as an auxiliary. First, I define (17) as an aggregate reaction function $K' = \Gamma(K; (k_{i,t}, a_{i,t+1})_i)$ for the case of homogeneous indivisibility $\lambda_i = \lambda$. $K$ enters $\Gamma$ via threshold $k^*_{i,t+1} = b_{i,t+1}K$, where $b_{i,t+1}$ is determined by (14). $\Gamma(K)$ represents the aggregation of in-

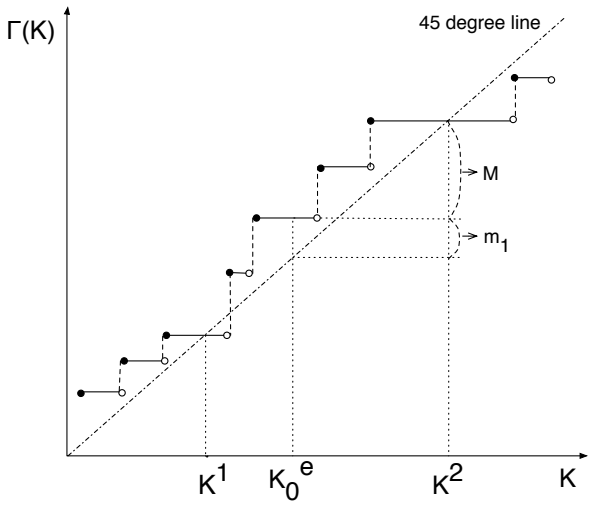


Figure 1: Aggregate reaction function $\Gamma$. $K^1$ is selected by Equilibrium Selection 1 since $|\log K^1 - \log K_0^e| < |\log K^2 - \log K_0^e|$. $K^2$ is selected by Equilibrium Selection 2 as $\text{sign}(\log K^2 - \log K_0^e) = \text{sign}(\log \Gamma(K_0^e) - \log K_0^e)$.

dividual capital when each firm optimally responds to aggregate capital $K$. Equilibrium aggregate capital is a fixed point of this reaction function. As depicted in Figure 1, $\Gamma$ is a non-decreasing step function.

Then, I define a new selection mechanism as follows:

**Equilibrium Selection 2** *For each initial capital vector $(k_{i,t})$, pick the equilibrium aggregate capital $K_{t+1}$ that attains the minimum of $|\log K_{t+1} - \log K_{t+1}^e|$ among all $K_{t+1}$ that solve (8, 17) and satisfy* $\text{sign}(\log K_{t+1} - \log K_{t+1}^e) = \text{sign}(\log \Gamma(K_{t+1}^e) - \log K_{t+1}^e)$.

This mechanism selects the equilibrium aggregate capital that is closest to the initial aggregate capital in the direction toward which the firms are

induced to adjust by the expected aggregate capital. In Figure 1, this mechanism selects $K^2$. Vives (1990) showed that the equilibrium selected by this mechanism can be reached as a convergent point of the best-response dynamics $K_{u+1} = \Gamma(K_u)$ starting at $K_0^e$. Cooper (1994) supported the use of this selection mechanism in macroeconomics on the grounds that the best-response dynamics is a realistic tatonnement process in a situation where many agents interact with each other. The only information needed for an agent to make decisions in the tatonnement is the aggregate capital level. Besides, this selection mechanism excludes the possibilities of big jumps that arise from a purely informational coordination among agents.

## 3.3 Homogeneity and extended policy function

I first analyze the fluctuation of the equilibrium selected by the second mechanism, and then proceed to analyze the one selected by the first mechanism. In this section, I concentrate on a homogeneous setup in which indivisibility and productivity are common across firms: $\lambda_i = \lambda$ and $a_{i,t} = 1$. Generalization to the case of heterogeneous indivisibility is discussed in Section 4.2. In this homogeneous setup, the only source of deviation from the expected aggregate capital is gap $s_{i,t}$. Model agents form expectations by the approximated gap distribution that is uniform over unit interval. Realization of a finite-length vector $(s_{i,t})_i$ necessarily deviates from the uniform distribution. This deviation from the uniform distribution corresponds to the deviation of the aggregate reaction function $\Gamma$ from the 45 degree line in Figure 1. This deviation is quite small when $N$ is large. Nonetheless, I show below that the difference between equilibrium aggregate capital and the expected one

persists even when $N$ tends to infinity.

With fixed productivity, an indivisible investment is induced only by natural depreciation and aggregate capital movements, and the threshold $s^*_{i,t}$ in (15) is simplified as $s^*_t = (\log K_{t+1} - \log K_t - \log(1-\delta))/\log\lambda$. Below, I work with an extended policy function $k^*_i = bK^\phi$ by introducing a new parameter $\phi \in (0, 1]$. The original policy function is a special case when $\phi = 1$. $\Gamma$ is redefined with this extended policy function, and $b$ is defined by (9) with homogeneity. The threshold for $s_{i,t}$ becomes

$$s^*_t = (\phi(\log K_{t+1} - \log K_t) - \log(1-\delta))/\log\lambda. \tag{24}$$

With $\phi = 1$, this corresponds to the original threshold (15). $\phi$ determines the strength of positive feedback from aggregate capital to individual investment decisions, and thus it represents the degree of strategic complementarity between investments. Introducing $\phi$ helps clarifying the role of complementarity in generating aggregate fluctuations in the following analysis.

## 3.4 Distribution of aggregate capital growth rate

The equilibrium aggregate capital growth rate, $\log K_{t+1} - \log K_t$, consists of an anticipated part $\log K^e_{t+1} - \log K_t$ and an unanticipated part $g_{t+1} \equiv \log K_{t+1} - \log K^e_{t+1}$. The former part is deterministic, since the expectation system determines $K^e_{t+1}$ given $K_t$. I focus on the distribution of the unanticipated growth $g_{t+1}$. I introduce a notation $q_t \equiv \log\lambda/(\phi(\log K^e_{t+1} - \log K_t) - \log(1-\delta))$. This is an inverse of the anticipated shift in $k^*_{t+1}$. At the steady state, $q_{ss}$ represents the natural frequency of a firm's capital adjustment. Henceforth, I drop the time subscript $t$ from all variables, and focus on $g$

given expected capital $K_0^e$.

Unanticipated growth $g$ is divided into two parts: adjustments in the initial round of fictitious tatonnement and subsequent adjustments. The first part, measured as the number of firms, is denoted as $m_1 \equiv N(\log \Gamma(K_0^e) - \log K_0^e)/\log \lambda$ (see Figure 1). If $m_1 = 0$, then $K^2 = K_0^e$ constitutes the equilibrium aggregate capital selected by mechanism 2. Otherwise, $K^2 \neq K_0^e$. I state the main technical result here.

**Proposition 4** *Under equilibrium selection 2, $Ng$ converges in distribution to $(m_1 + M)\log \lambda$, where $M$ conditional on $m_1 > 0$ follows:*

$$\Pr(M = w \mid m_1) = m_1 e^{-\phi(w+m_1)} \phi^w (w + m_1)^{w-1}/w! \tag{25}$$

*for $w = 0, 1, \ldots$. The unconditional distribution of $m_1 + M$ is symmetric. The tail of distribution (25) is approximated by*

$$\Pr(M = w \mid m_1) \sim (m_1 e^{(1-\phi)m_1}/\sqrt{2\pi}) e^{-(\phi - 1 - \log \phi)w} w^{-1.5}. \tag{26}$$

*$m_1/\sqrt{N}$ asymptotically follows a normal distribution with mean zero and variance $\sigma_1^2 = (1 - \lambda^{-2\rho/q})/(2\rho \log \lambda) - ((1 - \lambda^{-\rho/q})/(\rho \log \lambda))^2$.*

The proof is deferred to Appendix D. Here, I outline the proof. I use a fictitious tatonnement as a workhorse for characterizing the aggregate fluctuations. The fictitious tatonnement is defined by the best-response dynamics

of capital profile:

$$k_{i,1} = \begin{cases} \lambda(1-\delta)k_{i,0} & \text{if } (1-\delta)k_{i,0} < k^*_{i,0} \\ (1-\delta)k_{i,0} & \text{otherwise} \end{cases}, \tag{27}$$

$$k_{i,u+1} = \begin{cases} \lambda k_{i,u} & \text{if } k_{i,u} < k^*_{i,u} \\ k_{i,u}/\lambda & \text{if } k_{i,u} \geq \lambda k^*_{i,u} \\ k_{i,u} & \text{otherwise} \end{cases}, \tag{28}$$

where $K_u = \left(\sum_i k^\rho_{i,u}/N\right)^{1/\rho}$ and $k^*_{i,u} = bK^\phi_u$. Subscript $u$ represents a step in the fictitious tatonnement. Note that the best-response dynamics is consistent with the aggregate response function $K_{u+1} = \Gamma(K_u)$.

The expected number of firms that adjust capital in the first step is $N/q$. Their investments may not exactly balance with aggregate capital depreciation: $\Gamma(K^e_0)$ may not coincide with $K^e_0$. If not, the optimal threshold is updated under new aggregate capital and the adjustments in the second step take place. This procedure is iterated until there are no more firms that newly adjust.

Subsequent adjustments after the first step are measured in the number of firms that adjust capital upward in step $u$, denoted by $m_u$ for $u = 2, 3, \ldots, T$. If firms adjust downward (i.e., some firms that decide to invest in the first step, retract), $m_u$ is set as negative. Series $m_u$ are either positive or negative for all $u$ depending on $m_1 > 0$ or $m_1 < 0$. $M \equiv \sum_{u=2}^T m_u$ denotes the total number of firms that adjust capital subsequently after the first step of tatonnement. $T$ is the stopping time of the tatonnement: $T \equiv \min_{u:m_u=0} u$. The equilibrium capital vector is determined by the convergent point of the dynamics, $k_{i,T}$. $m_1 + M$ indicates the total deviation of the investment from the stationary level in terms of the number of firms.

In the first step toward Proposition 4, I show that the capital growth rate is asymptotically proportional to the number of firms that adjust.

**Lemma 1** *$N(\log K_{u+1} - \log K_u)$ converges to $m_{u+1} \log \lambda$ as $N \to \infty$ almost surely for $u = 1, 2, \ldots, T-1$.*

The proof is in Appendix B. Lemma 1 implies that $N(\log K^2 - \log K_0^e) \to (m_1 + M) \log \lambda$. Thus, the computation of $g$ reduces to counting the total number of investing firms net of the expected number of investing firms. I then show that the number of adjusting firms in the tatonnement asymptotically follows a Poisson branching process.

**Lemma 2** *$m_u$ for $u = 2, 3, \ldots, T$ asymptotically follows a branching process, in which each firm in $m_u$ bears firms in step $u+1$ whose number follows a Poisson distribution with mean $\phi$.*

The proof is in Appendix C. A branching process is an integer stochastic process of a population in which each parent in a generation bears a random number of children in the next generation. In a Poisson branching process, the number of children borne by a parent is a Poisson random variable. It is known that a branching process converges to 0 in a finite time period with probability 1 if the mean number of children borne by a parent is less than or equal to 1 (Feller, 1957, p.276). This fact confirms that the best-response dynamics stops in a finite time $T$ with probability 1 when $\phi \leq 1$. Thus, the best-response dynamics is a valid algorithm of equilibrium selection even when $N \to \infty$. Moreover, the cumulative population size of the Poisson branching process is known to follow a Borel-Tanner distribution (Kingman,

1993, p.68). By combining the Borel-Tanner distribution with the Poisson distribution for $m_2$, I obtain the desired distribution (25).

By Stirling's formula, the tail of (25) is approximated by (26). (26) shows that $g$ conditional on $m_1$ asymptotically follows a gamma-type distribution that combines a power function $w^{-1.5}$ and an exponential function $e^{-(\phi-1-\log\phi)w}$. Note that $\phi - 1 - \log\phi > 0$ for $\phi < 1$. Since an exponential function declines faster than a power function, the tail distribution is dominated by the exponential when $\phi < 1$. Thus, the degree of strategic complementarity $\phi$ determines the speed of the exponential truncation of the distribution.

$\phi = 1$ holds in the business cycles model in Section 2, and in this case, the distribution (26) becomes a power-law distribution with exponent 0.5. Whether the tail obeys an exponential decay or a power decay has an important implication for the moments of the distribution. If the tail decays exponentially, any $k$-th moment exists, because $\int_0^\infty x^k e^{-x} dx$ is a gamma function and thus finite. To the contrary, if the tail decays in power with exponent $\alpha$, only the moments lower than $\alpha$ exist, since $\int^\infty x^k x^{-\alpha-1} dx$ is finite only for $k < \alpha$. When the exponent of the power law is 0.5, even the mean diverges.

The macro-level fluctuation observed in the numerical simulations ensues from the criticality condition $\phi = 1$, which results in the power-law tail of the capital growth distribution. When this condition is not met, the aggregate fluctuations eventually die down as the number of firms increases to infinity. This is because $\phi$, the mean number of children per parent, determines the trend population growth in the branching process. The mean population of the $n$-th generation is $\phi^n$ given a single initial parent. The population

diverges to infinity when the process is supercritical, $\phi > 1$, whereas the population decreases to zero if subcritical, $\phi < 1$. At the critical point $\phi = 1$, the population of a generation decreases to zero with probability 1 and yet the mean cumulative population diverges to infinity.

## 3.5 Aggregate fluctuations with an arbitrarily large $N$

The distribution of $M$ conditional on $m_1$ follows a pure power-law distribution when $\phi = 1$. With exponent 0.5, the power-law distribution does not have either mean or variance. The conditioning variable $m_1$, which represents the initial deviation from expected capital in the tatonnement, obeys the law of large numbers and its variance decreases linearly in $N$. These two effects cancel out in the unconditional variance of $(m_1 + W)/N$, as I state in the following proposition.

**Proposition 5** *The variance of $g$ converges to a non-zero constant as $N \to \infty$ when $\phi = 1$. The limit standard deviation is* $(\log \lambda)\sqrt{(2/\pi)(\sigma_1 + 1/3)\sigma_1}$.

The proof is deferred to Appendix E. The main idea is as follows. Proposition 4 showed that $m_1/\sqrt{N}$ asymptotically follows a normal distribution with finite variance. This implies that the mean of absolute value $|m_1|$ scales as $\sqrt{N}$. Proposition 4 also showed that $Ng/\log\lambda - m_1$ conditional on $m_1 = 1$ follows a power-law distribution with exponent 0.5 if $\phi = 1$. Then, the variance of $Ng$ conditional on $m_1 = 1$ diverges as $N^{1.5}$, because $\int^N x^2 x^{-1.5} dx \sim N^{1.5}$. Combining these two results, I obtain that $Ng$ unconditional on $m_1$ has variance scaling as $N^2$, since $Ng$ can be divided into $\sqrt{N}$

sub-population sets, each of which has variance that scales as $N^{1.5}$. Hence, the variance of $g$ scales as $N^0$.

The argument above shows that the power-law distribution is essential in obtaining scale-invariant fluctuations for $g$. The key environment for the power law, $\phi = 1$, can be interpreted as perfect complementarity of indivisible investments. By perfect complementarity I mean that a proportional increase in capital of all the other firms induces the same proportional increase in the capital of a firm, if the increment is much larger than the indivisibility. Due to the indivisibility of capital, however, a shock smaller than indivisibility does not cause a symmetric movement across firms. Thus, the firm's investment behavior at criticality may be summarized as local inertia combined with global perfect complementarity.

It might appear counterintuitive that the aggregate variance does not converge to zero when there are only idiosyncratic discrepancies in the initial capital gap. Note that in a smoothly-adjusting, competitive economy, the aggregate capital level is indeterminate in the production sector if the firms' investment decisions are perfect complements due to the constant returns to scale technology. In the present model, the equilibrium is locally unique because of the indivisibility of capital. Nonetheless, the globally indeterminate environment makes it possible for the aggregate fluctuations to reappear in the form of the power-law distribution.

The limit standard deviation of $g$ in Proposition 5 is determined by indivisibility parameter $\lambda$ and periodicity $q$ of capital oscillation at the firm level. Numerical examples for the standard deviation shown in Table 3 suggest that an empirically plausible range of indivisibility can generate the magnitude of

| Markup ($\mu - 1$) | | 0.02 | | | | 0.2 | | | |
|---|---|---|---|---|---|---|---|---|---|
| Indivisibility ($\log \lambda$) | | 0.02 | 0.05 | 0.1 | 0.2 | 0.02 | 0.05 | 0.1 | 0.2 |
| | 4 | 0.92 | 2.29 | 4.56 | 9.07 | 0.92 | 2.29 | 4.57 | 9.10 |
| Periodicity ($q_{ss}$) | 6 | 0.82 | 2.04 | 4.07 | 8.11 | 0.82 | 2.04 | 4.08 | 8.13 |
| | 8 | 0.75 | 1.87 | 3.73 | 7.43 | 0.75 | 1.87 | 3.73 | 7.44 |

Table 3: Limit standard deviations of the aggregate capital growth rate (percent)

fluctuations observed in the frequency of business cycles.

I also note that the indivisibility parameter $\log \lambda$ has an almost proportional effect on the aggregate standard deviation when periodicity $q$ is held constant. This is because the fluctuation magnitude shows little dependence on the markup rate. In fact, the standard deviation does not significantly change even when the markup rate goes to infinity, at which $\sigma_1^2$ is simplified to $(1 - 1/q)/q$. The proportional impact of $\log \lambda$ on the aggregate standard deviation can be also seen in numerical simulation, as in Table 2. This implies that the indivisibility of capital provides a foundation for the sizable idiosyncratic volatility of the firm-level decisions, which in turn has one-to-one impact on the aggregate volatility.

## 3.6 Equilibrium selection as in the simulation

Finally, I investigate the magnitude of the fluctuation in aggregate capital selected by equilibrium selection 1. Let $g^1$ denote the capital growth rate under this selection. I obtain the following result.

**Proposition 6** *Under equilibrium selection 1, the convergence of the variance of* $g^1$ *to zero as* $N \to \infty$ *is not faster than* $1/\sqrt{N}$ *if* $\phi = 1$.

Proof is in Appendix F. Proposition 6 shows that if I choose the least-volatile equilibrium, the variance of the capital growth rate decreases to zero as $N$ increases, but at a rate much slower than what the central limit theorem predicts. In the Long-Plosser model with continuous capital adjustments, idiosyncratic technological shocks cancel out in aggregation and the aggregate variance declines as fast as $1/N$ (Dupor (1999)). In contrast, in this model, the variance of $g^1$ declines no faster than $1/\sqrt{N}$. This again opens up a theoretical possibility that indivisible investment at the micro level contributes to sizable macro-level fluctuations when the number of firms is large but finite.

Table 2 shows the standard deviations in the numerical simulations of equilibrium paths when the number of firms is 100000, about one-third of the benchmark case. Standard deviations exhibit no discernible increase from the benchmark. Thus, in the calibrated range of parameter values, the number of firms does not have a significant impact on the magnitude of aggregate fluctuations.

# 4 Discussion

## 4.1 Model assumptions and information content of gap distribution

Assumptions important for aggregate fluctuations are capital indivisibility and predetermined prices of goods. If indivisibility $\log \lambda$ is 0, Proposition 5

states that the asymptotic aggregate variance is 0. This is because, under continuous adjustments, capital keeps track of idiosyncratic productivity, which cancels out across firms. The predetermined price is important for the criticality condition $\phi = 1$. Under flexible prices of goods, an increase in aggregate investment raises factor prices and dampens the investment demand as in Thomas (2002). In contrast, under fixed prices, an increase in aggregate investment causes firms to hire more, which raises consumption and prevents interest rate increases.

In the model, I also assume that firms use the stationary gap distribution in order to form expectations. In deciding $k_{t+1}$, firms observe $K_{t+1}$ and $X_t$ but not $X_{t+1}$. To form expectations on $w_{t+1}$ and $R_{t+1}$, firms assume that future gap distributions are uniformly distributed. This assumption is necessary to render the model tractable. Simulated gap distributions show little deviation from the uniform distribution. In what follows, I further examine whether using the exact gap distribution significantly improves firms' prediction power over future prices.

The exact gap distribution $s_{i,t}$ enters the equilibrium system (17, 18, 19) in summations directly and in information sets of conditional expectations. The former effect is negligible. The difference between the summations in (19) when evaluated by exact gaps and by the integral over the stationary distribution is less than $10^{-13}$ percent in the benchmark simulation. As $N$ becomes larger, this error becomes even smaller. The latter effect by the gap distribution in information sets might be more potent. However, fully implementing this state variable raises the curse of dimensionality in numerical computation. Krusell and Smith, Jr. (1998) deal with this by

transforming the distribution equivalently to an infinite vector of moments and then approximating it by a finite vector. In this model, I approximate the gap distribution by its stationary counterpart in a continuum economy. In what follows, I check that the gap distribution does not have significantly higher prediction power for future prices than the stationary distribution.

For each period $t$, the expectation for investment threshold $s^*_{i,t}$ is formed as (15). The expected investment is the integral of indivisible investments of firms with $s_{i,t}$ less than $s^*_{i,t}$. Actual investment may differ from the expected value due to two factors: the exact values of $s_{i,t}$ and the realizations of idiosyncratic productivities. If the exact $s_{i,t}$ is known for some future period $t$ and no productivity shocks are present, then the distribution of $s_{i,t}$ should have prediction power for the difference between the expected and realized investments. This prediction power is weakened when the idiosyncratic shocks wash out the information $s_{i,t}$ has. To see this, I conduct the following experiment. In a variation of the benchmark simulation, I compute the equilibrium path selected by mechanism 2. I additionally compute $\log \Gamma(K^e_{t+1}) - \log K^e_{t+1}$ for the case where productivity shocks in $t+1$ are set at 0. This log-difference and $\log K_{t+1} - \log K^e_{t+1}$ turn out to exhibit no correlation either in terms of values or signs. This implies that exact $s_{i,t}$ alone cannot even predict the sign of expectation error because of the uncertain productivities. This experiment confirms that the exact gap distribution has little prediction power for future prices even at the small standard deviation (0.05%) of idiosyncratic productivity shocks, because aggregate investments respond to the small perturbations in productivity or capital gap non-linearly and quite sensitively.

## 4.2 Heterogeneity, criticality, and power law

In the previous section, the fluctuation distribution is derived under the assumption of homogeneous capital indivisibility. However, empirical studies attest large variations in the lumpiness in investment-capital ratio across firms (Doms and Dunne (1998); Cooper, Haltiwanger, and Power (1999)). In Appendix G, I show that the power-law tail distribution with the same exponent is obtained even in the general setup where the indivisibility and depreciation rates are heterogeneous across firms. The robustness of the exponent 0.5 results from the fact that *any* branching process with martingale property brings out the power-law tail with exponent 0.5 for the cumulative population size (Harris, 1989, p.32).[5][6] When random productivity is incorporated, it is possible to characterize the fluctuation in the form of a moment generating function (Nirei (2006)), but it becomes difficult to derive the distribution function analytically.

The possibility of a power-law distribution of sectoral propagation was first pointed out by Bak, Chen, Scheinkman, and Woodford (1993). In a simple supply chain model, they obtained a power-law distribution of aggregate fluctuations with exponent 1/3. The difference in exponent arises from

[5]The distribution of population size in branching processes is closely related to the distribution of the first return time of a random walk, which has the same power-law exponent 0.5.

[6]The robustness reflects the fact that in various models of connected non-linear dynamics, the perfect complementarity $\phi = 1$ appears as a condition for idiosyncratic shocks to have aggregate consequences through power-law distributions. For example, in a celebrated theorem by Erdős and Rényi, the condition $\phi = 1$ corresponds to the critical point for the emergence of a "giant cluster" in a random graph (Bollobás, 1998, p.240).

the topology of the model network. They assumed a two-dimensional lattice network in which two avalanches starting from neighboring sites can overlap. This leads to a longer chain reaction and a flatter power-law tail. In contrast, the present model features a market equilibrium that is essentially dimensionless in terms of the firms' network. The market model corresponds to an infinite-dimension case of the lattice models, which yields the cluster-volume exponent 0.5 (Grimmett, 1999, p.256).

The power-law tail is useful in understanding the mechanism for aggregate fluctuations that arise from idiosyncratic shocks. It allows a slowly-aggregating mechanism such as in the case of equilibrium selection 1 to generate the magnitude of fluctuations comparable to business cycles with a realistically large number of firms. However, the power law is not easily testable in the business cycles data. $M|m_1 = 1$, the number of firms induced to invest by a single investing firm, is not observable. The number of investing firms $M$ is observable, and follows a mixture of the power law for $M|m_1$ and the normal distribution for $m_1$. In simulations, the unconditional distribution of $M$ is best fitted by a Laplace-like distribution with exponentially declining tails. Thus, the aggregate fluctuation distribution reflects some traits of the non-Gaussian distribution but not the pure power law.

## 4.3 Fragile equilibria, coordination failure, and fiscal policy implications

I argued that perfect complementarity with discrete choice generates aggregate fluctuations of a locally unique equilibrium in a globally indeterminate environment. This can be translated as jumpy dynamics of a ball on a flat,

rugged landscape in the language of "fragile equilibria" models by Blanchard and Summers (1988). In that context, the contribution of this paper is to derive the distribution of the jumps in a standard business cycle model. This model also features a coordination failure in the sense of Cooper and John (1988). The coordination failure occurs in the capital market where households delegate investment decisions to firms. Since households only instruct discount factors to firms while firms face non-convexity (indivisibility), inefficient allocations may arise in equilibrium. This inefficiency could be arbitraged by, for example, a financial intermediary who finances investments of multiple firms and delivers smoother dividends to households. Such intermediaries are abstracted from the model. This setup can be justified when it is costly to collect exact information on firms' capital positions. In the model economy, none of the agents possesses such an informational advantage, and further, the possible gain of arbitrage by smoothing consumption is quantitatively very small.

As a predetermined pricing model that entails quantity adjustments upon demand shocks, this model has an implication for active fiscal policies. It is possible to include government purchase in the model and show that the fiscal stimulus generates a multiplier effect in the same mechanism as an investment demand shock propagating to consumption and output. However, implementing an active demand control for stabilization would require the government to have information as much as the potential financial arbitrager discussed above, and would be impractical in this model. Moreover, potential welfare gains by such stabilization policy would be as small as the profits of the potential financial arbitrager in the current model with representative

households. To fruitfully discuss the stabilization policy, the model needs to incorporate heterogeneous households, government's capability to identify investment shocks, and realistic labor and financial markets. Moreover, the current model does not generate strong enough autocorrelations in key variables. Extensions of this model toward a full account of business cycles and stabilization policies are left for future research.

# 5 Conclusion

This paper characterizes the aggregate fluctuations arising from the complementarity of indivisible investments at the firm level. Analytically, I propose to evaluate the fluctuation of aggregate investment along the evolution of heterogeneous capital as if it is a stochastic fluctuation whose randomness arises from the stochastic configuration of relative capital levels. For each configuration, the equilibrium aggregate investment is determined as a convergent point of a fictitious best-response dynamics of firms' investment decisions. The best-response dynamics can be embedded in a branching process with a probability measure of the stochastic configuration of relative capital. This enables us to derive the distribution function of the aggregate fluctuation in a closed form.

The fluctuation in the number of investing firms is shown to follow a power-law distribution with an exponential truncation at the tail. The truncation speed is determined by the degree of strategic complementarity among firms. In the model of predetermined price setting with constant returns to scale technology, the distribution becomes a pure power law, and the standard

deviation of the growth rate is shown to be strictly positive even when there are an infinite number of firms. The limiting standard deviation is shown to be almost proportional to the indivisibility of firm-level investments.

I incorporate the above fluctuation mechanism in a dynamic general equilibrium model and numerically compute equilibrium paths without making the randomness assumption of the capital configuration. Under plausible parameter values, the equilibrium path is shown to exhibit aggregate fluctuations comparable to business cycles in magnitude and correlation structure. The simulation also confirms the validity of the analysis above that utilizes the assumptions of randomness and uniformity of the capital configuration.

# A Proof of Proposition 1

The right-hand side of gap dynamics in (10) is written as a modulo 1 of

$$\frac{\log(1-\delta)+\log(a_t^0 K_t)-\log(a_{t+1}^0 K_{t+1})+\frac{\rho}{\alpha(1-\rho)}(\log a_{i,t}-\log a_{i,t+1})}{\log\lambda_i}+s_{i,t}+1,$$

where

$$a_t^0 \equiv \left(\int a_{i,t}^{\frac{\rho}{\alpha(1-\rho)}}\left(\frac{\lambda_i^\rho-1}{\lambda_i-1}\right)^{\frac{\rho}{1-\rho}}\lambda_i^{\rho s_{i,t}}di\right)^{\frac{-1}{\rho}}. \tag{29}$$

$s_{i,t}$ is obtained by applying a modulo 1 operation to the sum of $t\log(1-\delta)/\log\lambda_i$, $(\log(a_0^0K_0)-\log(a_t^0K_t)+(\rho/(\alpha(1-\rho)))(\log a_{i,0}-\log a_{i,t}))/\log\lambda_i$, and $s_{i,0}+1$. When $1/\log\lambda_i$ has a well-defined density, the first term, taken modulo 1, converges in distribution to a unit uniform random variable as $t\to\infty$, and its sum with an absolutely continuous random variable, taken modulo 1, also converges to the unit uniform distribution (Engel, 1992, pp.28-29). Since the second and third terms also have densities, they are absolutely

continuous and satisfy the condition of this theorem.

# B Proof of Lemma 1

Let $H_u$, $u = 2, 3, \ldots, T$ denote the set of firms that adjust capital in step $u$. Assume that the size of $H_u$ is finite with probability one when $N \to \infty$, which I verify later. I consider the case $m_1 > 0$ for the proofs of Lemmas 1 and 2 and Proposition 4 without loss of generality. Thus, $\log k_{i,u} = \log k_{i,u-1} + \log \lambda$ for $i \in H_u$.

The Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ around $(\log k_u)_{i \in H_{u+1}}$ is calculated as follows. The first derivative is $\partial N \log K_u / \partial \log k_{i,u} = (k_{i,u}/K_u)^\rho$. Thus, $\partial K_u / \partial k_{i,u}$ is of order $1/N$.[7] The second and higher derivatives with respect to own $\log k_{i,u}$ are $\partial^n (k_{i,u}/K_u)^\rho / \partial \log k_{i,u}^n = \rho^n (k_{i,u}/K_u)^\rho + O(\partial K_u / \partial k_{i,u})$ for $n = 1, 2, \ldots$. The second cross derivatives, $\partial^2 \log K_u / (\partial \log k_{i,u} \partial \log k_{j,u})$, are of order $\partial K_u / \partial k_{j,u}$, and thus, $O(1/N)$. Similarly, the higher-order cross derivative terms with respect to the capital of $h$ distinct firms in $H_{u+1}$ are of order $1/N^{h-1}$. Since $H_{u+1}$ is finite, the $n$-th derivative of $N \log K_u$ has a finite number of cross derivative terms for any finite $n$. Hence, the Taylor series expansion of $N(\log K_{u+1} - \log K_u)$ yields

$$\sum_{n=1}^{\infty} \sum_{i \in H_{u+1}} \left( \frac{k_{i,u}}{K_u} \right)^\rho \frac{\rho^{n-1} (\log \lambda)^n}{n!} + O(1/N) = \frac{\lambda^\rho - 1}{\rho} \sum_{i \in H_{u+1}} \left( \frac{k_{i,u}}{K_u} \right)^\rho + O(1/N),$$

where I used $\lambda^\rho = \lambda^0 + \sum_{n=1}^{\infty} (d^n \lambda^\rho / d\rho^n)|_{\rho=0} (\rho^n / n!)$. Utilizing $k_{i,u} = k_u^* \lambda^{s_{i,u}}$, I obtain that $\sum_{i \in H_{u+1}} (k_{i,u}/K_u)^\rho = (\sum_{i \in H_{u+1}} \lambda^{s_{i,u}\rho}) / (\sum_{i=1}^{N} \lambda^{s_{i,u}\rho} / N)$. The

[7] By taking $y_N$ as of order $x_N$, or interchangeably, $y_N = O(x_N)$, I mean that $y_N / x_N$ converges to a finite number as $N \to \infty$.

denominator converges to $\mathrm{E}[\lambda^{s_{i,u}\rho}]$ as $N \to \infty$ almost surely by the law of large numbers, and I have $\mathrm{E}[\lambda^{s_{i,u}\rho}] = \int_0^1 \lambda^{s_{i,u}\rho} ds_{i,u} = (\lambda^\rho - 1)/(\rho \log \lambda)$. The numerator, $\sum_{i \in H_{u+1}} \lambda^{s_{i,u}\rho}$, converges to $m_{u+1}$ for every event when $H_{u+1}$ is finite, because $s_{i,u}$ is smaller than $\phi(\log K_u - \log K_{u-1})/\log \lambda$ for any $i \in H_{u+1}$, and thus, $\lambda^{s_{i,u}}$ converges to 1 as $N \to \infty$. Thus, I obtain the lemma.

# C Proof of Lemma 2

The conditional probability for firm $i$ to invest in $u = 2, 3, \ldots, T$ is

$$\Pr(i \in H_u \mid i \notin \cup_{v=2,3,\ldots,u-1} H_v) = \frac{\phi(\log K_u - \log K_{u-1})/\log \lambda}{1 - \phi(\log K_{u-1} - \log K_0)/\log \lambda}. \quad (30)$$

Thus, $m_u$ follows a binomial distribution with population $N - \sum_{v=2}^{u-1} m_v$ and probability (30). The mean of $m_u$ converges to $\phi m_{u-1}$ as $N \to \infty$, by using Lemma 1. Then, the binomial distribution of $m_u$ converges to a Poisson distribution with mean $\phi m_{u-1}$ for $u = 2, 3, \ldots, T$. Since a Poisson distribution is infinitely divisible, the Poisson variable with mean $\phi m_{u-1}$ is equivalent to a $m_{u-1}$-times convolution of a Poisson variable with mean $\phi$. Thus, the process $m_u$ for $u = 2, 3, \ldots, T$ is a branching process with a Poisson random variable with mean $\phi$, where $m_2$ follows a Poisson distribution with mean $\phi m_1$. Note that $m_1$ is not included in the branching process because it is not necessarily an integer.

# D Proof of Proposition 4

The initial state of the best-response dynamics (27, 28) is constructed as follows. For the case $\phi < 1$, a gap profile $(s_{i,0})_i$ is drawn from a jointly

uniform distribution, and a capital profile is constructed by $k_{i,-1} = \lambda^{s_{i,0}} K_{-1}^{\phi}$ and $K_{-1} = (\sum_{i=1}^{N} k_{i,-1}^{\rho}/N)^{1/\rho}$. For $\phi = 1$, the last equation determines $b_{-1}$ and $B_{-1}$ given the realized gaps, and $K_{-1}$ is determined by $B_{-1}$ in the expectation system. Then, the initial state of the tatonnement is constructed by $k_{i,0} = (1-\delta)k_{i,-1}$ and by setting $K_0$ at $K_0^e$ that is determined by the expectation system given $K_{-1}$. In the notations for tatonnement, $\log K_T - \log K_{-1}$ corresponds to the capital growth rate in the model $\log K_{t+1} - \log K_t$.

I first derive the asymptotic distribution of $M$ conditional on $m_1$. It is known that the accumulated sum $M = \sum_{u=2}^{T} m_u$ of the Poisson branching process conditional on $m_2$ follows an infinitely divisible distribution called the Borel-Tanner distribution (Kingman, 1993, p.68):

$$\Pr(M = w \mid m_2) = (m_2/w)e^{-\phi w}(\phi w)^{w-m_2}/(w-m_2)! \tag{31}$$

for $w = m_2, m_2+1, \ldots$. By combining (31) with $m_2$ that follows the Poisson distribution with mean $\phi m_1$, and using the binomial theorem in the summation over $m_2$, I obtain (25).[8] Furthermore, the approximation in (26) is obtained by applying Stirling's formula $w! \sim \sqrt{2\pi}e^{-w}w^{w+0.5}$ and the fact that $(1+m_1/w)^{w-1} \to e^{m_1}$ as $w \to \infty$.

Next, I derive the asymptotic normal distribution of $m_1/\sqrt{N}$. I split $m_1/\sqrt{N}$ into three terms as $\log\Gamma(K_0) - \log(\sum_{i=1}^{N}((1-\delta)k_{i,-1})^{\rho}/N)^{1/\rho}$, $\log(\sum_{i=1}^{N}((1-\delta)k_{i,-1})^{\rho}/N)^{1/\rho} - \log K_{-1}$, and $\log K_{-1} - \log K_0$, all multiplied by $\sqrt{N}/\log\lambda$. The second term represents the depreciation and is equal to $(\sqrt{N}/\log\lambda)\log(1-\delta)$. Thus, the sum of the second and third terms yields $-\sqrt{N}/q$. The first term represents the first-step adjustments induced directly by depreciation. Define $H_1$ as the set of firms that adjust in the first step. Using $k_{i,0} = \lambda^{s_{i,0}} k_0^*$,

[8] See the Technical Appendix for a detailed derivation.

I obtain $K_1 = (1-\delta)k_0^*((\lambda^\rho - 1)\sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N + \sum_{i=1}^N \lambda^{s_{i,0}\rho}/N)^{1/\rho}$ and $(\sum_{i=1}^N((1-\delta)k_{i,0})^\rho/N)^{1/\rho} = (1-\delta)k_0^*(\sum_{i=1}^N \lambda^{s_{i,0}\rho}/N)^{1/\rho}$. Hence, the first term of $m_1/\sqrt{N}$ becomes

$$\frac{\sqrt{N}}{\rho\log\lambda}\log\left((\lambda^\rho-1)\frac{\sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N}{\sum_{i=1}^N\lambda^{s_{i,0}\rho}/N}+1\right). \tag{32}$$

By assumption, $s_{i,0}$ is distributed uniformly. Thus, the denominator $\sum_{i=1}^N \lambda^{s_{i,0}\rho}/N$ in (32) converges to $\int_0^1 \lambda^{s_{i,0}\rho}ds_{i,0} = (\lambda^\rho-1)/(\rho\log\lambda)$ with probability one by the law of large numbers. Let $x$ denote the numerator: $x \equiv \sum_{i\in H_1}\lambda^{s_{i,0}\rho}/N$. Note that $i \in H_1$ is equivalent to $0 \le s_{i,0} < 1/q$. Then, the asymptotic mean of $x$ is $x_0 = \int_0^{1/q}\lambda^{s_{i,0}\rho}ds_{i,0} = (\lambda^{\rho/q}-1)/(\rho\log\lambda)$, and by the central limit theorem, $\sqrt{N}(x-x_0)$ converges in distribution to the normal distribution with mean zero and variance

$$\int_0^{1/q}\left(\lambda^{s_{i,0}\rho}\right)^2 ds_{i,0} - \left(\frac{\lambda^{\rho/q}-1}{\rho\log\lambda}\right)^2 = \frac{\lambda^{2\rho/q}-1}{2\rho\log\lambda} - \left(\frac{\lambda^{\rho/q}-1}{\rho\log\lambda}\right)^2. \tag{33}$$

I regard (32) as a function $F$ of $x$. By the delta method, I obtain that $F(x)$ asymptotically follows the normal distribution with mean $F(x_0)$ and variance $F'(x_0)^2\mathrm{Avar}(x)$. $F(x_0)$ is calculated as

$$\frac{\sqrt{N}}{\rho\log\lambda}\log\left(\frac{(\lambda^\rho-1)(\lambda^{\rho/q}-1)/(\rho\log\lambda)}{(\lambda^\rho-1)/(\rho\log\lambda)}+1\right) = \frac{\sqrt{N}}{q}. \tag{34}$$

This cancels out with the second and third terms of $m_1/\sqrt{N}$. $F'(x_0)^2\mathrm{Avar}(x)$ is calculated as $\sigma_1^2$ in the proposition. Then, $m_1/\sqrt{N}$ asymptotically follows a normal distribution with mean zero and variance $\sigma_1^2$. This completes the proof.

# E Proof of Proposition 5

Lemma 1 implies that $(\log K^2 - \log K_0^e)/\log\lambda$ asymptotes to $(m_1 + M)/N$, which I focus on here. Its unconditional variance $\mathrm{Var}\left((m_1 + M)/N\right)$ is decomposed as follows:

$$\begin{aligned}
&\mathrm{E}\left[\mathrm{Var}\left(\frac{M}{N} \mid m_1\right)\right] + \mathrm{Var}\left(\frac{m_1}{N} + \mathrm{E}\left[\frac{M}{N} \mid m_1\right]\right) \qquad (35)\\
&= \mathrm{E}\left[\mathrm{E}\left[\mathrm{Var}\left(\frac{M}{N} \mid m_1, m_2\right) \mid m_1\right] + \mathrm{Var}\left(\mathrm{E}\left[\frac{M}{N} \mid m_1, m_2\right] \mid m_1\right)\right]\\
&+ \mathrm{Var}\left(\frac{m_1}{N} + \mathrm{E}\left[\mathrm{E}\left[\frac{M}{N} \mid m_1, m_2\right] \mid m_1\right]\right).
\end{aligned}$$

$m_u$ asymptotically follows a martingale branching process when $N \to \infty$ and $\phi = 1$. Thus, $|M|$ conditional on $|m_2|$ is asymptotically equivalent to the $|m_2|$-times convolution of $M$ conditional on $m_2 = 1$. Using these facts, I obtain that

$$\mathrm{Var}(M/N \mid m_1, m_2) \sim |m_2|\mathrm{Var}(M/N \mid m_2 = 1), \qquad (36)$$

$$\begin{aligned}
\mathrm{E}[\mathrm{E}[M/N \mid m_1, m_2] \mid m_1] &\sim \mathrm{E}[m_2 \mid m_1]\mathrm{E}[M/N \mid m_2 = 1] \qquad (37)\\
&\sim m_1\mathrm{E}[M/N \mid m_2 = 1].
\end{aligned}$$

Further, $|m_2|$ conditional on $m_1$ asymptotically follows a Poisson distribution with mean $|m_1|$, and the unconditional distribution of $m_2$ is symmetric. Since $m_1/\sqrt{N}$ asymptotically follows $N(0, \sigma_1^2)$ by Proposition 4, I can use

the formula $\mathrm{E}[|m_1|/\sqrt{N}] \to \sigma_1\sqrt{2/\pi}$. Applying these, I obtain

$$\begin{aligned}
&\mathrm{Var}\left(\frac{m_1+M}{N}\right) \qquad (38)\\
&\sim \mathrm{E}\left[\mathrm{E}\left[|m_2| \mid m_1\right]\mathrm{Var}\left(\frac{M}{N} \mid m_2=1\right)+\mathrm{Var}(m_2 \mid m_1)\mathrm{E}\left[\frac{M}{N} \mid m_2=1\right]^2\right]\\
&+\mathrm{Var}\left(\frac{m_1}{N}+\mathrm{E}\left[\frac{M}{N} \mid m_2=1\right]\mathrm{E}[m_2 \mid m_1]\right)\\
&\sim (\sigma_1\sqrt{2/\pi})\mathrm{E}\left[\frac{M^2}{N^{1.5}} \mid m_2=1\right]+\sigma_1^2\left(\frac{1}{\sqrt{N}}+\mathrm{E}\left[\frac{M}{\sqrt{N}} \mid m_2=1\right]\right)^2.
\end{aligned}$$

Next, I calculate $\lim_{N\to\infty}\mathrm{E}[M/\sqrt{N} \mid m_2=1]$. By Proposition 4, the probability of $M=N$ declines as $N^{-1.5}$ when $\phi=1$. Since $N\cdot N^{-1.5}$ converges to 0, the conditional expectation of $M$ can be evaluated by using the asymptotic probability function (31) when $\phi=1$:

$$\Pr(M=w \mid m_2=1)=e^{-w}w^{w-1}/w!. \qquad (39)$$

By using inequality (Feller, 1957, p.52)

$$\sqrt{2\pi}w^{w+0.5}e^{-w+1/(12w+1)}<w!<\sqrt{2\pi}w^{w+0.5}e^{-w+1/(12w)}, \qquad (40)$$

the upper and lower bounds of the asymptotic mean of $M/\sqrt{N}$ are computed as follows:[9]

$$\sum_{w=1}^{N}e^{-w}w^w/(w!\sqrt{N})<\int_0^N w^{-0.5}dw/\sqrt{2\pi N}\to\sqrt{2/\pi}, \qquad (41)$$

$$\sum_{w=1}^{N}e^{-w}w^w/(w!\sqrt{N})>\int_1^{N+1}e^{-1/(12w)}w^{-0.5}dw/\sqrt{2\pi N}\to\sqrt{2/\pi}. \qquad (42)$$

Hence, $\mathrm{E}[M/\sqrt{N} \mid m_2=1, W\le N]\to\sqrt{2/\pi}$. Similarly, I obtain

$$\mathrm{E}[M^2/N^{1.5} \mid m_2=1]\to 1/(1.5\sqrt{2\pi}). \qquad (43)$$

[9]See the Technical Appendix for a detailed derivation.

Collecting the results, I obtain $\mathrm{Var}((m+M)/N) \to (2/\pi)(\sigma_1+1/3)\sigma_1$. Hence, the capital growth rate has an asymptotic variance: $(\log\lambda)^2(2/\pi)(\sigma_1+1/3)\sigma_1$.

# F Proof of Proposition 6

Consider the case $\Gamma(K_0^e) > K_0^e$ depicted in Figure 1. $K^1$ is the fixed point of $\Gamma$ on the opposite side of $K_0^e$ from $K^2$. There exists a point between $K^1$ and $K_0^e$ at which $\Gamma$ crosses the 45 degree line from below. By applying Proposition 4, the number of adjusting firms between the point and $K^1$ follows the power law with exponent 0.5 if $\phi = 1$. Then, the tail distribution of $|\log K^1 - \log K_0^e|$ cannot decay faster than the power function with exponent 0.5.

By the selection rule, $|g^1| = \min\{|\log K^1 - \log K_0^e|, |\log K^2 - \log K_0^e|\}$. Since the two terms in the minimization operator are independent conditional on $m_1$, I have that $\Pr(|g^1| > g \mid m_1) = \Pr(|\log K^1 - \log K_0^e| > g \mid m_1)\Pr(|\log K^2 - \log K_0^e| > g \mid m_1)$. Thus, $g^1$ conditional on $m_1$ has a tail that cannot decay faster than the power function with exponent $0.5+0.5 = 1$. At the power exponent 1, the variance of $g^1$ conditional on $m_1$ decreases as $\int^N x^2 x^{-2} dx/N^2 \sim 1/N$. Since the mean of $|m_1|$ increases as $\sqrt{N}$, proceeding as in the proof of Proposition 5, I obtain that the variance of $g^1$ decreases as $1/\sqrt{N}$. If the tail distribution of $g^1$ conditional on $m_1$ decays more slowly than the power law with exponent 1, the variance of $g^1$ also decreases more slowly than $1/\sqrt{N}$.

# G Heterogeneous indivisibility

In this section, I extend the fluctuation results to the case where both the indivisibility and depreciation rates are heterogeneous across firms. Suppose that there are a finite $L$ types of firms with parameter values $\delta_i = \delta(l)$ and $\lambda_i = \lambda(l)$ for $l = 1, 2, \ldots, L$. Each type of firm is drawn with probability $\sigma(l)$, where $\sum_{l=1}^{L} \sigma(l) = 1$. The lower bound of the inaction band becomes heterogeneous as in (8): $k_i^* = b_i K^{\phi}$. I maintain that productivity $a_{i,t}$ is homogeneous across $i$. Define $\log \check{\lambda}_i \equiv b_i^{\rho}(\lambda_i^{\rho} - 1)/(\rho \mathrm{E}[b_i^{\rho} \lambda_i^{s_{i,1}\rho}])$. Then, I obtain the following proposition.

**Proposition 7** *Suppose that $\lambda_i$ and $\delta_i$ vary across firms, and are randomly drawn from a finite set. Then, $M$ conditional on $m_1 = 1$ follows the same tail distribution as (26):*

$$\Pr(|M| = w \mid m_1 = 1) = C_0 (e^{\check{\phi}-1}/\check{\phi})^{-w} w^{-1.5} \tag{44}$$

*for a large integer $w$, where $\check{\phi} \equiv \phi \mathrm{E}[\log \check{\lambda}_i / \log \lambda_i]$ and $C_0$ are constant. The asymptotic variance of the fraction of firms that adjust, $(m_1 + M)/N$, is strictly positive when $N \to \infty$ if $\check{\phi} = 1$.*

Proof: Let $N(l)$ denote the total number of firms of type $l$ and $m_u(l)$ denote the number of firms of type $l$ that adjust capital in step $u$.

First, I show the counterpart of Lemma 1 as follows:

$$N(\log K_{u+1} - \log K_u) = \sum_{n=1}^{\infty} \sum_{i \in H_{u+1}} \left(\frac{k_{i,u}}{K_u}\right)^{\rho} \frac{\rho^{n-1}(\log \lambda_i)^n}{n!} + O(1/N) \tag{45}$$
$$= \frac{\sum_{i \in H_{u+1}} b_i^{\rho} \lambda_i^{s_{i,u}\rho} \sum_{n=1}^{\infty} \rho^{n-1}(\log \lambda_i)^n / n!}{\sum_i b_i^{\rho} \lambda_i^{s_{i,u}\rho} / N} + O(1/N)$$
$$\to \frac{\sum_{i \in H_{u+1}} b_i^{\rho}(\lambda_i^{\rho} - 1)/\rho}{\mathrm{E}[b_i^{\rho} \lambda_i^{s_{i,u}\rho}]}.$$

Define $Z_{u+1}$ as the right-hand side of (45). It has mean $m_{u+1}\mathrm{E}[\log \check{\lambda}_i]$.

I then show that $(m_u)_u$ follows a branching process. Let $F_1$ denote the cumulative distribution function of $s_{i,1}$. I then have

$$\Pr(i \in H_u, \lambda_i = \lambda(l) | i \notin \cup_{v=2,3,\ldots,u-1} H_v) \tag{46}$$
$$= \sigma(l) \frac{F_1(\phi(\log K_u - \log K_0)/\log \lambda(l)) - F_1(\phi(\log K_{u-1} - \log K_0)/\log \lambda(l))}{1 - F_1(\phi(\log K_{u-1} - \log K_0)/\log \lambda(l))}.$$

Thus, $m_u(l)$ follows a binomial distribution with probability as stated above and population $N(l) - \sum_{v=2}^{u-1} m_v(l)$. Considering that $m_v$'s, $v = 2, 3, \ldots, u-1$ are finite with probability one, I obtain the asymptotic mean of the binomial as $\sigma(l)\phi Z_u / \log \lambda(l)$. Thus, $m_u(l)$ asymptotically follows a Poisson distribution with this mean. Hence, $m_u = \sum_{l=1}^{L} m_u(l)$ asymptotically follows a Poisson distribution with mean $\phi \mathrm{E}[\log \check{\lambda}_i / \log \lambda_i] m_{u-1} = \check{\phi} m_{u-1}$.

The vector of Poisson random variables $(m_u(l))_l$ conditional on its sum $m_u$ follows a multinomial distribution with probability vector $((\sigma(l)/\log \lambda(l))/\mathrm{E}[1/\log \lambda_i])_l$ and population $m_u$ (Kingman, 1993, page 7). $Z_u$ is the sum of a multinomial vector with weights $b(l)^{\rho}(\lambda(l)^{\rho} - 1)/(\rho \mathrm{E}[b_i^{\rho} \lambda_i^{s_{i,u}\rho}])$. Thus, $Z_u$ conditional on $m_u$ is equivalent to a $m_u$-times convolution of a random variable. Then, $m_{u+1}$ conditional on $m_u$ asymptotically follows a compound Poisson distribution. Since a compound Poisson distribution is infinitely divisible, $(m_u)_u$ follows a

branching process in which each firm in step $u$ bears children in step $u+1$ whose number follows the compound Poisson distribution that has mean $\check{\phi}$. By the theorem by Otter, a cumulative sum of a branching process follows the distribution as in Proposition (Harris, 1989, p.32). Finally, the process $(m_u)$ is finite with probability 1 if $\check{\phi} \leq 1$. This completes the proof.

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